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THE FIRST BOUNDARY VALUE PROBLEMS WITHOUT INITIAL CONDITIONS FOR SCHRÖDINGER SYSTEMS IN NONSMOOTH DOMAIN

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INTRODUCTION

HISTORY AND REASONS FOR SELECTING TOPICS

The boundary value problems for differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology. In pure mathematics, these equations are studied from several different perspectives, mostly concerned with their solutions - the set of functions that satisfy the equation. When we consider these equations, initial conditions are often the main reasons to ensure the uniqueness of solutions. However, there are many differential equations which we can not give the initial conditions. Problems without initial conditions for evolution equations arise when describing different nonstationary processes in nature. We consider so a distant initial time that the initial condition practically has no influence on present time, while boundary conditions do affect it. So we can assume that the initial time is $t = -\infty$. We will try to describe here the main methods for solving problems without initial conditions related to evolution equations, grounding on model equations (see N. M. Bokalo, A. Lorenzi (2009)).

Consider the following problem related to the heat equation and subject to the Dirichlet boundary condition:

$$u_t(t,x) - \Delta u(t,x) = f(t,x),\ (t,x) \in S \times \Omega, \tag{1}$$

$$u(t,x) = h(t,x),\ (t,x) \in S \times \partial\Omega, \tag{2}$$

where Ω is a domain in $\mathbb{R}^n$ with a piecewise-smooth boundary $\partial\Omega$, $S = (-\infty, 0]$ or $S = \mathbb{R}$, and f, h are given functions. Obviously, when $f = 0, h = 0$ and $\Omega = (0, \pi)$ or $\Omega = (0, +\infty)$ the functions

$$u_C(t,x) = Ce^{-t} \sin x,\ (t,x) \in S \times \Omega,$$

$C \in \mathbb{R}$ being an arbitrary constant, are classical solutions of problem (1)-(2). So, to ensure the uniqueness of the solution of this

problem, it is necessary to impose an additional condition on the solution. By the way, the standard initial condition does not suit, because the initial moment is $t = -\infty$. We note that such a problem was first solved by A. N. Tihonov. In his work, he showed that, when $\Omega = (0, \pi)$, for the uniqueness of the solution of the problem, it is enough to require that it should be bounded on $S \times \Omega$. To prove this result, Tihonov used the integral representation of the solution of the first initial boundary value problem by the corresponding Green function. In the same way, the (unique) bounded solution of problem (1)-(2), with $f = 0$, admits the representation:

$$u(t,x) = \frac{1}{2\sqrt{\pi}} \int\limits_{-\infty}^{t} \frac{x}{(t-\tau)^{3/2}} \exp\{\frac{-x^2}{4(t-\tau)}\} h(\tau) d\tau,$$

for all $(t, x) \in S \times [0, +\infty)$, under the assumption that the function h be continuous and bounded on S. A.N. Tihonov named such a problem the *Fourier problem* hay *the problem without initial conditions* for the heat equation. Later on, A.N. Tihonov's ideas were used in the works of T. M. Balabushenko and S. D. Ivasyshen (2002), S. D. Ivasyshen (1983) devoted to solving problems without initial conditions for general linear parabolic systems with boundary conditions different from the Dirichlet one. The integral expression of the solution of such problems was obtained and well-posedness solvability theorems were proved in local Hölder spaces when solutions are bounded and increasing functions.

There is another approach to solve problems without initial conditions for a few evolution equations. For the sake of simplicity we will show it for the heat equation when $S = (-\infty, 0]$. For this purpose we need to introduce the following additional conditions on the classical solution of (1)-(2).

$$\sup_{(t,x)\in S\times\overline{\Omega}} |u(t,x)| < +\infty, \quad \sup_{x\in\overline{\Omega}} |u(t,x)| \to 0 \text{ when } t \to -\infty. \quad (3)$$

Making use of the maximum principle, it is possible to prove the uniqueness of the solution of the problem (1)-(2) subject to con-

ditions (3). The proof of the existence of classical solutions to this problem, relies on cut-off functions and an "exhaustion" method for the interval S. Modifications of the previous method can be used for solving problems without initial conditions for different classes of evolution equations, in particular, quasi-linear and with time delay ones. In this case a different behavior at $t \to -\infty$ can be allowed, involving also unbounded solutions.

Let us now consider the case of generalized solutions. Note that when dealing with generalized solutions for problems without initial conditions related to possibly nonlinear parabolic equations different approaches are needed according whether domain Ω is bounded or not. When the domain Ω is bounded, two cases may occur:

1) to ensure uniqueness and existence of the solution additional conditions are needed involving both the behaviour as t to ∞ of the solution u and the right-hand side f .

2) no such conditions are needed.

First we consider problem (1)-(2) when the spatial domain Ω is bounded and $h = 0$. We assume that function f admits the representation

$$f(t,x) = f_0 + \sum_{l=0}^{n} D_{x_l} f_l(t,x), \ (t,x) \in S \times \Omega, \tag{4}$$

where $\{f_0, \ldots, f_n\} \subset L^2_{loc}(S \times \overline{\Omega})$ and the derivatives are to be understood in the sense of distributions $\mathcal{D}'(S \times \Omega)$. To ensure the uniqueness of the solution, we need some additional requirements on the behavior of the solution u as $t \to -\infty$

$$e^{\lambda_1 t}|u(t,x)| \to 0 \text{ as } t \to -\infty; \tag{5}$$

and to ensure the existence of the solution, the right hand-side f

have to satisfy the following condition

$$\int\limits_S e^{-2\omega t}||f||^2_{H^{-1}(\Omega)}dt < \infty, \tag{6}$$

where $\omega > -\lambda_1$ and λ_1 be the first eigenvalue of the operator $-\Delta$ with the first boundary condition. Results of this type for different classes of evolution equations and associated variational inequalities can be found in the works of N. M. Bokalo (1994), V. M. Dmytriv (2000), Yu. B. Dmytryshyn (2009), G. P. Domans'ka, M. O. Kolin'ko, S. P. Lavrenyuk (2006), Z. Hu (2005), A. A. Pankov (1985), P. Ya. Pukach (1994), ...

Let us now consider problem without initial conditions when the domain Ω is not bounded. The results for some classes of evolution equations are given in the paper of S. P. Lavrenyuk, N. Protsakh (2007). We quote also that problems without initial conditions for evolution equations involving second-order in time derivatives, linear Sobolev Hal'pern systems, hyperbolic-type and nonclassical equations are studied in the following authors' works V. M. Kirilich, A. D. Myshkis (1992), S. P. Lavrenyuk, M. O. Kolin'ko (1998), S. P. Lavrenyuk, N. Protsakh (2007), S. P. Lavrenyuk, M. A. Oliskevich (2014), D. Safarov (1990), G. O. Vafodorova (2000, 2003), ...

To sum up, there exists many works considering the problems without initial conditions. Continue the above direction of study, there are some open problems:

- Other properties of generalized solution.
- Problems without initial conditions for other evolution classes systems.
- The domain Ω contains singular points.

Many important applied problems reduce to the study of boundary value problems for partial differential equations in nonsmooth

domains. These questions have been extensively discussed in the literature of the fundamental work of V. A. Kondratiev (1967). By now the theory of boundary-value problems for elliptic equations in non-smooth domains has been worked out in much detail, with a large literature on it. Parallel with this theory, the boundary-value problems for non-stationnary equations and systems have been studied by many athors, such as G. Eskin (1992), A. Yu. Kokotov and B. A. Plamenevskii (2005), ... In these works, they used results and methods of elliptic boundary-value problems in nonsmooth domains to prove the assertions on the unique solvability, on the smoothness and the asymptotic expansions of solutions near the singularities on the boundary.

Schrödinger equations and systems play important role in partial differential equations theory because of these applications in quantum mechanics. Boundary value problems for Schrödinger equations have been considered in the books by Lions and Magenes (1968) in finite cylinders $Q_T = \Omega \times (0, T),\ T < +\infty$ with base Ω, where $\partial\Omega$ is smooth. Their results are restricted to Schrödinger type equations, where coefficients a_{pq} of equations are functions independent of t (except a_{00}). The first initial boundary value problem for general Schrödinger systems, where the coefficients $a_{pq}(x, t)$ are matrices of functions of two variables x and t for all p, q, was considered in papers of N. M. Hung (1998), in cases the finite cylinder $Q_T,\ T < +\infty$ or in cases the infinite cylinder $Q_\infty = Q \times (0, \infty)$ as in N. M. Hung and C. T. Anh (2004). The such second boundary value problems was investigated by N. M. Hung and N. T. K. Son (2008). In these works, to consider asymptotic behaviour of solutions in a neighborhood of conical points, authors just studied the case when the associated spectrum problem has simple eigenvalues or has semisimple eigenvalues having invariant multiplicities.

In this thesis, we consider the first boundary value problems without initial conditions for Schrödinger systems in nonsmooth

domain. We study the solvability, the regularity of solutions and asymptotic formulas of solutions near the conical points.

THE AIM OF THESIS

The main aim of the thesis is to study the solvability, the regularity of solutions and asymptotic formulas of solutions of the problem without initial condition in domains with singular points.

METHOD OF THE THESIS

The main methods used in the thesis are traditional techniques of problems without initial conditions and boundary value problems in domain with singular points, such as:

- To prove the existence of generalized solutions of the problems without initial conditions: firstly, we construct approximating solutions u^h of problems with the initial conditions $t = h$. Then, by letting $h \to -\infty$, the solvability of problem without initial condition is obtained.

- To the studying of the uniqueness solvability of the problem without initial conditions: Ladyzenskaya's method to choose the test functions.

- To consider the existence of generalized solutions of initial-conditions the problems: Galerkin's approximating method

- To prove regularity of solutions of the problem: the induction method.

- To investigate the asymptotic behaviour of the solutions near the singular points: we transfer our problem to elliptic boundary-value problems depending on a real parameter in domains with conical points.

STRUCTURE AND RESULTS OF THESIS

Without the introduction, inclusion, author's works related to the thesis that have been published and references, the thesis consists of three chapters.

- Chapter 1: The uniqueness solvability of the problem.
- Chapter 2: The regularity of solution of the problem.
- Chapter 3: The asymptotic formulas of solution of the problem near the conical points.

Chapter 1

THE UNIQUENESS SOLVABILITY OF THE PROBLEM

In this chapter, we consider the boundary value problem without initial condition for Schrödinger systems in cylinders. Firstly, we prove the existence of solutions u^h of problems with the initial conditions $t = h$. Then, by letting $h \to -\infty$, the solvability of problem without initial condition is obtained. The main result in this chapter is the Theorem 1.2. The contents of this chapter is written based on the paper 1, 2 in the works of author related to the thesis that has been published.

1.1. Setting problem

We now introduce a $2m$- order differential operator

$$L(x,t,D) = \sum_{|p|,|q|=0}^{m} (-1)^{|p|} D^p\big(a_{pq}(x,t)D^q\big),$$

where a_{pq} are $s \times s$ matrices of functions which are mesureable and bounded on $\overline{Q}$ and $a_{pq} = a^*_{qp}$ for $|p| = |q| = m$ (a^*_{qp} denotes the transposed conjugate matrix of a_{pq}). Suppose that there exists a possitive constant a_0 such that for all $\xi \in \mathbb{R}^n \setminus \{0\}$, $\eta \in \mathbb{C}^s \setminus \{0\}$ and $(x,t) \in Q$, the following estimate holds

$$\sum_{|p|=|q|=m} a_{pq}(x,t)\xi^p\xi^q\eta\overline{\eta} \geq a_0|\xi|^{2m}|\eta|^2,$$

where $\xi^p = \xi_1^{p_1}...\xi_n^{p_n}$, $\xi^q = \xi_1^{q_1}...\xi_n^{q_n}$.

In this thesis, we consider the following problem in the cylinder Q:

$$(-1)^{m-1}iL(x,t,D)u - u_t = f(x,t) \text{ in } Q, \tag{1.1}$$

$$\frac{\partial^j u}{\partial \nu^j}\,|_\Gamma = 0, \quad j = 0,...,m-1, \tag{1.2}$$

where ν is the unit vector of outer normal to the surrounding surface Γ.

1.2. The unique solvability of problems with initial condition.

Firstly, for any $h \in \mathbb{R}$ we study the following problem in the cylinder Ω_h^∞:

$$(-1)^{m-1} iL(x,t,D)v - v_t = f(x,t) \text{ in } \Omega_h^\infty, \tag{1.3}$$

$$v \mid_{t=h} = 0,\ x \in \Omega, \tag{1.4}$$

$$\frac{\partial^j v}{\partial \nu^j} \mid_{S_h^\infty} = 0, \quad j = 0, ..., m-1, \tag{1.5}$$

where ν is the unit vector of outer normal to the surrounding surface S_h^∞.
The unique existence of this problem can be formulated as follows.

Theorem 1.1. *Assume that*

$$i,\ \sup\left\{ \left|\frac{\partial a_{pq}}{\partial t}\right| : (x,t) \in \Omega_h^\infty,\ 0 \le |p|, |q| \le m \right\} = \mu < \infty;$$

$$ii,\ f_t, f \in L_2(-\gamma, \Omega_h^\infty).$$

Then for all $\gamma > \gamma_0 = \dfrac{m^\star \mu}{2\mu_0}$, $m^\star = \sum\limits_{|\alpha| \le m} 1$, *there exists uniquely generalized solution* $v \in \overset{\circ}{H}{}^{m,0}(-\gamma, \Omega_h^\infty)$ *of problem* (1.3)-(1.5) *sastisfying:*

$$\|v\|^2_{H^{m,0}(-\gamma,\Omega_h^\infty)} \le C\left[\|f\|^2_{L_2(-\gamma,\Omega_h^\infty)} + \|f_t\|^2_{L_2(-\gamma,\Omega_h^\infty)} \right], \tag{1.6}$$

where C *is nonnegative constant independent of* h, v *and* f.

1.3. The uniqueness of generalized solution of problem without initial conditions

In this section, we consider the Dirichlet boundary value problems without initial for Schrödinger systems. The main result on the solvability of the problems can be formulated as follows.

Theorem 1.2. *Assume that:*

$$i,\ \sup\left\{|\frac{\partial a_{pq}}{\partial t}| : (x,t)\in Q,\ 0\le |p|,|q|\le m\right\} = \mu < \infty;$$

$$ii,\ |\frac{\partial a_{pq}}{\partial t}| = o(e^{2\gamma t})\ \textit{as}\ t\to -\infty\ \textit{for all}\ |p|,|q|\le m\ \textit{and}\ x\in\Omega;$$

$$iii,\ f, f_t \in L_2(-\gamma, Q).$$

Then for all $\gamma > \gamma_0 = \dfrac{m^\star \mu}{2\mu_0}$, $m^\star = \sum\limits_{|\alpha|\le m} 1$, *there exists uniquely generalized solution* $u(x,t) \in \overset{\circ}{H}{}^{m,0}(-\gamma, Q)$ *of problem* (1.1)-(1.2) *sastisfying:*

$$\|u\|^2_{\overset{\circ}{H}{}^{m,0}(-\gamma,Q)} \le C\left[\|f\|^2_{L_2(-\gamma,Q)} + \|f_t\|^2_{L_2(-\gamma,Q)}\right].$$

Chapter 2

THE REGULARITY OF SOLUTION OF THE PROBLEM

Our main aim here is to study the regularity of solution with respect to time variable and spatial variables. First, we give results on the regularity of problems (1.3)-(1.5) with the initial condition $t = h$. Then, because of the fact that the constants C in the priori estimates do not depend on the initial condition $t = h$, by letting $h \to \infty$, the smoothness of generalized solution of our problem is obtained. The main results in this chapter are Theorem 2.3 and Theorem 2.4 The contents of this chapter is written based on the paper 2 in the works of author related to the thesis that has been published.

2.1. The regularity with respect to time variable of the problem with initial condition.

By using induction, the smoothness of solution with respect to time variable of the problem (1.3)-(1.5) can be stated as follows.

Theorem 2.1. *Assume that $l \in \mathbb{N}$ and exist $\mu > 0$ such that*

$$i,\ \sup\left\{ |\frac{\partial a_{pq}}{\partial t}| : (x,t) \in Q,\ 0 \le |p|, |q| \le m, \right\} = \mu < \infty,$$
$$|\frac{\partial^k a_{pq}}{\partial t^k}| \le \mu_2,\ \mu_2 > 0,\ \ \textit{for } 2 \le k \le l+1;$$

$$ii,\ f_{t^k} \in L_2(-\gamma, \Omega_h^\infty),\ \ \textit{for } k \le l+1,\ f_{t^k}(x,h) = 0 \textit{ for } x \in \Omega,$$
$$0 \le k \le l.$$

Then for all $\gamma > (2l+1)\gamma_0$, the solution $v \in \overset{\circ}{H}{}^{m,0}(-\gamma, Q)$ of problem (1.3)-(1.5) *has derivatives with respect to t up to order l*

satisfying $v_{t^k} \in H^{m,0}(-\gamma, \Omega_h^\infty)$, *for all* $k = 0, \dots, l$ *such that*

$$\|v_{t^k}\|^2_{H^{m,0}(-\gamma,\Omega_h^\infty)} \leq C \sum_{j=0}^{k+1} \|f_{t^j}\|^2_{L_2(-\gamma,\Omega_h^\infty)}, \tag{2.1}$$

where the constant C *does not depend on* v, f, h.

2.2. The regularity with respect to both time and spatial variable of the problem with initial condition

Assume that ω is a local coordinate system on S^{n-1}. And more assume that the principal part of the operator $L(x,t,D)$ at origin 0 can be written in the form

$$\mathcal{L}(0,t,D) = r^{-2m}\mathfrak{L}(\omega, t, rD_r, D_\omega),\ D_r = \frac{i\partial}{\partial r},$$

where $\mathfrak{L}$ is a linear operator with smooth coefficients. Consider the following spectral problem

$$\mathfrak{L}(\omega, t, \lambda, D_\omega)v(\omega) = 0,\ \omega \in G, \tag{2.2}$$

$$D^j_\omega v(\omega) = 0,\ \omega \in \partial G,\ j = 0, ..., m-1. \tag{2.3}$$

Theorem 2.2. *Let* l *be a nonnegative integer and* γ *be a real number such that* $\gamma > (2(2m+l)+1)\gamma_0$. *Assume that* $v(x,t)$ *is a weak solution in the space* $H^{m,0}(-\gamma, \Omega_h^\infty)$ *of the problem* (1.3)-(1.5) *and* $f_{t^k} \in H_0^{l,0}(-\gamma, \Omega_h^\infty)$, *for* $k \leq 2m+l+1$, $f_{t^k}(x,h) = 0$ *for* $k \leq 2m+l$. *In addition, suppose that in the trip*

$$m - \frac{n}{2} \leq Im\lambda \leq 2m + l - \frac{n}{2}$$

there is no point from the spectrum of the equation (2.2)-(2.3). *Then we have* $v \in H_0^{2m+l}(-\gamma, \Omega_h^\infty)$ *and the following estimate holds*

$$\|v\|^2_{H_0^{2m+l}(-\gamma,\Omega_h^\infty)} \leq C \sum_{k=0}^{2m+l+1} \|f_{t^k}\|^2_{H_0^{l,0}(-\gamma,\Omega_h^\infty)}. \tag{2.4}$$

2.3. The regularity of solution of the problems without initial conditions

In this section, using properties of approximate solutions u^h of the problem (1.3)-(1.5), we obtain the following results on the smoothness of solution of problem (1.1)-(1.2).

Theorem 2.3. *Assume that $l \in \mathbb{N}$ and exists $\mu > 0$ such that*

$$i,\ \sup\left\{|\frac{\partial a_{pq}}{\partial t}| : (x,t) \in Q,\ 0 \le |p|, |q| \le m, \right\} = \mu < \infty,$$

$$|\frac{\partial^k a_{pq}}{\partial t^k}| \le \mu_2,\ \mu_2 > 0,\ \ \textit{for } 2 \le k \le l+1;$$

$$ii,\ f_{t^k} \in L_2(-\gamma, Q),\ \textit{for all } x \in \Omega,\ 0 \le k \le l+1.$$

Then for all $\gamma > (2l+1)\gamma_0$, the solution $u \in \overset{\circ}{H}{}^{m,0}(-\gamma, Q)$ of problem (1.1)-(1.2) has derivatives with respect to t up to order l satisfying $u_{t^k} \in H^{m,0}(-\gamma, Q)$, $\forall k = 0, \dots, l$ such that

$$\|u_{t^k}\|^2_{H^{m,0}(-\gamma,Q)} \le C \sum_{j=0}^{k+1} \|f_{t^j}\|^2_{L_2(-\gamma,Q)},$$

where the constant C does not depend on u, f.

Theorem 2.4. *Let l be a nonnegative integer and γ be a real number such that $\gamma > (2(2m+l)+1)\gamma_0$. Assume that $u(x,t)$ is a weak solution in the space $H^{m,0}(-\gamma, Q)$ with of the problem (1.1)-(1.2) and $f_{t^k} \in H_0^{l,0}(-\gamma, Q)$, for $k \le 2m+l+1$. In addition, suppose that in the trip*

$$m - \frac{n}{2} \le Im\lambda \le 2m + l - \frac{n}{2}$$

there is no point from the spectrum of the equation (2.2) - (2.3). Then we have $u \in H_0^{2m+l}(-\gamma, Q)$ and the following estimate holds

$$\|u\|^2_{H_0^{2m+l}(-\gamma,Q)} \le C \sum_{k=0}^{2m+l+1} \|f_{t^k}\|^2_{H_0^{l,0}(-\gamma,Q)}.$$

Chapter 3

THE ASYMPTOTIC FORMULAS OF SOLUTION OF THE PROBLEM NEAR THE CONICAL POINTS

In this chapter, we establish several results on the asymptotic behaviour of solutions near singular points. The main Theorem is stated in Theorem 3.1. To obtain these results, we use the representation of the solutions to the elliptic boundary value problems near conical points and induction method. The contents of this chapter is written based on the paper 3 in the author's works related to the thesis that has been published.

3.1. Preliminaries

This section is devoted to contruct the pencil operator

$$\mathfrak{U}(\lambda, t) = (\mathfrak{L}(\omega, t, \lambda, D_\omega), D_\omega^0, ..., D_\omega^{m-1}),\ \lambda \in \mathbb{C},\ t \in \mathbb{R},$$

of the following parameter - depending elliptic systems

$$\mathfrak{L}(\omega, t, \lambda, D_\omega)u = f, \text{ in } G, \tag{3.1}$$

$$D_\omega^{j-1}u = g_j, \text{ on } \partial G,\ j = 1, ..., m. \tag{3.2}$$

To receive asymptotic formulas of the solutions with the coefficients regular with respect to time variable, we require later that the eigenvalues and eigenvectors of the pencil $\mathfrak{U}(\lambda, t)$ satisfy the following assumption.
Let l_1, l_2 be nonnegative integers and β_1, β_2 be real numbers such that $l_1 - \beta_1 < l_2 - \beta_2$. We say that the assumption (**H**) for numbers $l_1, l_2, \beta_1, \beta_2$ is fulfilled if the following conditions are satisfied.

i. The lines $\text{Im}\ \lambda = -\beta_i + l_i - n/2$, $(i = 1, 2)$ do not contain eigenvalues of the pencil $\mathfrak{U}(\lambda, t)$, and the strip $-\beta_1 + l_1 - n/2 < \text{Im}\ \lambda < -\beta_2 + l_2 - n/2$ contains eigenvalues $\lambda_\mu(t)$, $\mu = 1, ..., N$, with the geometric multiplicities Λ_μ and the partial multiplicities $\kappa_{\mu,k}$, $\mu = 1, ..., N$, $k = 1, .., \Lambda_\mu$, not

depending on $t \in \mathbb{R}$. These eigenvalues are analytic functions on $\mathbb{R}$.

ii. A canonical system $\varphi_{k,s}^{(\mu)}(\omega, t)$, $k = 1, ..., \Lambda_\mu$, $s = 1, ..., \kappa_{\mu,k}$, of Jordan chains of $\mathfrak{U}(\lambda, \mathfrak{t})$ corresponding to the eigenvalues $\lambda_\mu(t)$, $\mu = 1, ..., N$ can be chosen, which consists of smooth functions on $\mathbb{R}$ for all $\omega \in \overline{G}$.

3.2. The asymptotic behaviour of solution of the elliptic system depending on a parameter

In this section, we study elliptic boundary-value problems, depending on a real parameter, in domains with conical points. We present asymptotic formulas for solutions near singular points, as linear combinations of special singular functions and regular functions. These functions and the coefficients of the linear combination are regular with respect to the parameter.

Lemma 3.1. *Let $u \in V_{\beta_1,d}^{l_1,0}(-\gamma, K_\infty)$ be a solution of the following problem*

$$(-1)^m L(t, D)u = f \ \ in\ K_\infty, \tag{3.3}$$

$$\frac{\partial^{j-1} u}{\partial \nu^{j-1}} = g_j \ \ on\ \partial K_\infty,\ j = 1, ..., m, \tag{3.4}$$

*in which $f \in V_{\beta_2,d}^{l_2-2m,0}(-\gamma, K_\infty)$, $g_j \in V_{\beta_2,d}^{l_2-2m+j+1/2,0}(-\gamma, \partial K_\infty)$, for $j = 1, ...m$ and l_1, l_2, d are nonnegative integes such that $l_2 \geq l_1 \geq 2m$, β_1, β_2 are real numbers that satisfy $l_1 - \beta_1 < l_2 - \beta_2$. Moreover, we assume that the assumption (**H**) is fulfilled for l_1, l_2, β_1, β_2 with $N = 1$. Then the solution u has the form*

$$u(x, t) = \sum_{s=0}^{\kappa-1} r^{-i\lambda_0(t)+s} P_s(\ln r) + w(x, t), \tag{3.5}$$

where $w(\cdot, \cdot) \in V_{\beta_2,d}^{l_2,0}(-\gamma, K_\infty)$, $P_s(\cdot)$ are vectors of polynomials of order less than $\theta + \kappa - 1$, whose coefficients are functions

in space $C^{\infty,d}(-\gamma, G_\infty)$, κ *is the minimum integer greater than* $-\beta_2 + l_2 - n/2 - Im\ \lambda_0(t)$ *for all* $t \in \mathbb{R}$, *and* θ *is the maximal partial multiplicity of the eigenvalue* $\lambda_0(t)$.

3.3. The asymptotic formulas of solutions to the boundary value problem without initial condition for Schrödinger systems in domain with conical points

The main result can be formulated as following.

Theorem 3.1. *Let* l, h *be nonnegative integers, let* β, β' *be real numbers satisfying* $0 \le \beta' < \beta$ *and* $\beta \ge m$. *Assume that* u *in* $\overset{\circ}{H}{}^{m,0}(-\gamma, Q)$ *be a generalized solution of problem* (1.1)-(1.2) *with* $\gamma > (2(d+l)+1)\gamma_0$; $f_{t^k} \in H_0^{l,0}(-\gamma, Q)$, *for* $k \le d + 2l + 2$ *and the assumption* (***H***) *for* $l_1 = 2m$, $l_2 = 2m + l$, $\beta_1 = \beta$, $\beta_2 = \beta'$ *is fulfilled. Suppose moreover that for all* $T > 0$ *the following conditions are satisfied:*

i. $Im\ \lambda_1(t) < ... < Im\ \lambda_N(t)$, $t \in [-T, T]$;

ii. $-\beta + 2m - n/2 < Im\ \lambda_1(t) < -\beta + 2m - n/2 + \mu_1^\star < Im\ \lambda_2(t) < ... < -\beta + 2m - n/2 + \mu_{N-1}^\star < Im\ \lambda_N(t) < -\beta' + 2m + l - n/2$, $t \in [-\infty, -T) \cup (T, \infty]$;

iii. $Im\ \lambda_j(t) \ne Im\ \lambda_k(t) + z$, $z \in \mathbb{Z}$, $j \ne k \in \{1, ..., N\}$, $t \in \mathbb{R}$,

where $\mu_j^\star$, $j = 1, ..., N-1$ *are nonnegative numbers. Then the following representation holds:*

$$u(x,t) = \sum_{j=1}^{N} \sum_{s=0}^{l+\kappa_j-1} r^{-i\lambda_j(t)+s} P_{s,j}(\ln r) + w(x,t), \tag{3.6}$$

where $w(\cdot,\cdot) \in V_{\beta',d+l}^{2m+l,0}(-\gamma, Q)$, $P_{s,j}(\cdot)$ *are vectors of polynomials of order less than* $l(\theta_j + 2) + \kappa_j + \theta_j - 1$, *whose coefficients are functions in space* $C^{\infty,d+l}(-\gamma, G_\infty)$, κ_j *is the minimum integer greater than* $-\beta' + 2m - n/2 - Im\ \lambda_j(t)$ *for all* $t \in \mathbb{R}$, $j = 1, ..., N$

and θ_j is the maximal partial multiplicity of the eigenvalue $\lambda_j(t)$ for $j = 1, ..., N$.

3.4. Examples

3.4.1. Example 1

Let $K = \{x = (x_1, x_2) \in \mathbb{R}^2 : r > 0,\ 0 < \omega < \omega_0\}$, where (r, ω) are polar coordinate of point $x = (x_1, x_2)$ and $0 < \omega_0 < 2\pi$.

Assume that $S^0 = \{x \in \mathbb{R}^2 : r > 0, \omega = 0\}$, $S^1 = \{x \in \mathbb{R}^2 : r > 0, \omega = \omega_0\}$, $S^j_\infty = S^j \times \mathbb{R}$, $j = 0, 1$. For any multi-index number $\alpha = (\alpha_1, \alpha_2)$, denote $\alpha_1 + \alpha_2 = |\alpha|$ and $\partial^\alpha_x = \partial^{|\alpha|}/\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}$.

This section is devoted to consider the following problem.

$$iLu - u_t = f \text{ in } K_\infty, \tag{3.7}$$

with the Dirichlet boundary condition

$$u\,|_{S^j_\infty} = 0,\ j = 0, 1, \tag{3.8}$$

where $L = L(x, t, \partial_x) = \sum_{j,k=1}^2 a_{jk}\partial_{x_j}\partial_{x_k}$ be a second-order partial differential operator. Moreover, suppose that $a_{jk}(t) = a_{kj}(t)$ are continuous in $x \in \Omega$ uniformly with respect to $t \in \mathbb{R}$ and

$$\sum_{j,k=1}^{2} a_{jk}(t)\eta_j\overline{\eta_k} \geq \rho_1|\eta|^2, \tag{3.9}$$

for all $\eta = (\eta_1, \eta_2) \in \mathbb{C}^2$, $t \in \mathbb{R}$, where ρ_1 be a possitive constant. Assume that the principal part of L near the origin O admits the following representation

$$\mathcal{L}(t, \partial_x) = \sum_{j,k=1}^{2} a_{jk}\partial_{x_j}\partial_{x_k}.$$

By the condition (3.9), there exists a possitive constant ρ_2 such that $a_{jj}(t) \geq \rho_2$ for all $t \in \mathbb{R}$. So, we can suppose that $a_{22}(t) \equiv 1$,

then $\mathcal{L}$ can be rewritten as following

$$\mathcal{L} = (\partial_{x_2} - a(t)\partial_{x_1})(\partial_{x_2} - \overline{a(t)}\partial_{x_1}), \tag{3.10}$$

where $a(t) = \alpha(t) + i\beta(t)$, $\alpha(t)$, $\beta(t)$ be real-value functions on $\mathbb{R}$, $\beta(t) \geq \rho_3$ for all t and ρ_3 be a possitive constant. Rewrite the problen (3.1)-(3.2) as

$$\mathcal{L}(\omega, t, \lambda, D_\omega)u = f, \text{ in } K_\infty, \tag{3.11}$$

$$u \mid_{\omega=0} = 0, \tag{3.12}$$

$$u \mid_{\omega=\omega_0} = 0, \tag{3.13}$$

where

$$\mathcal{L}(\omega, t, \lambda, D_\omega) = \Big((-\sin\omega + a(t)\cos\omega)i\lambda + (\cos\omega + a(t)\sin\omega)\partial_\omega\Big) \Big((-\sin\omega + \overline{a(t)}\cos\omega)i\lambda + (\cos\omega + \overline{a(t)}\sin\omega)\partial_\omega\Big). \tag{3.14}$$

The pencil operator corresponding to the problem (3.11)-(3.13) is defined as follows.

$$\mathfrak{U}(\lambda, t) = (\mathcal{L}(\omega, t, \lambda, D_\omega), 1, 1).$$

Proposition 3.1. *Let $u(x,t)$ be a generalized solution of the problem (3.7)-(3.8) where the differential operator L admits the representation (3.10), and $a(t)$ be an analytic function. Moreover, let $f_{t^k} \in H_0^{l,0}(-\gamma, Q)$ for $k \leq d$. Assume that the lines $Im\lambda = 0$, $Im\lambda = 1$ does not contain any eigenvalue of the operator $\mathfrak{U}$ and in the strip $0 < Im\lambda < 1$, there are simple eigenvalues $\lambda_1(t), \ldots, \lambda_N(t)$ of the problem (3.11)-(3.13) for all t. Then, we have*

$$u(x,t) = \sum_{j=1}^{N} \sum_{s=0}^{\kappa_j - 1} r^{-i\lambda_j(t)+s} P_{s,j}(\ln r) + w(x,t), \tag{3.15}$$

where $w(\cdot,\cdot) \in V^{2,0}_{0,d}(-\gamma, Q)$, $P_{s,j}$ *are vectors of polynomials of order less than* κ_j, *whose coefficients are functions in space* $C^{\infty,d}(-\gamma, G_\infty)$, *and* κ_j *is the minimum integer greater than* $1 - Im\lambda_j(t)$ *for all* $t \in \mathbb{R}$, $j = 1, \ldots, N$.

3.4.2. Example 2

In this section, we give an example where all eigenvalues of the pencil operator $\mathfrak{U}$ depending on the time variable t.

Let $\omega_0 \in (\frac{3\pi}{2}, 2\pi), \alpha(t) \equiv 0, \beta(t) = \dfrac{\tan \dfrac{\omega_0}{1 + \epsilon\omega_0 \arctan \sigma(t)}}{\tan \omega_0}$, where $\epsilon > 0$ small enough such that $\beta(t) \geq \rho_3 > 0$ and $\sigma(t)$ satisfied $|a'(t)| = o(e^{2\gamma t})$ as $t \to -\infty$. Applying Theorem 3.1, we get the following results.

Proposition 3.2. *Assume that* $f_{t^k} \in L_2(-\gamma, \mathbb{R})$ *for* $k \leq d+2$ *and let* u *be a generalized solution of the problem (3.7)-(3.8). Then, the following fomular holds*

$$u(x,t) = cr^{\pi/\omega_0 + \pi\epsilon \arctan \sigma(t)} + u_1(x,t),$$

where $u_1 \in V^{2,0}_{0,d}(-\gamma, Q)$ *and* $c \in C^{\infty,d}(-\gamma, G_\infty)$.

3.4.3. Example 3

In this example we consider the problem for the Schrödinger equation in quantum mechanics:

$$i\Delta u - u_t = f \text{ in } Q, \tag{3.16}$$

with the Dirichlet boundary condition

$$u\,|_\Gamma = 0. \tag{3.17}$$

Proposition 3.3. *Let* $f_{t^k} \in L_2(-\gamma, \mathbb{R})$ *for* $0 \leq k \leq 3$ *and let* u *be a generalized solution of the problem* (3.16)-(3.17) *for* $n = 2$. *Then we have*

- *if* $\omega_0 < \pi$ *then* $u \in H_0^2(-\gamma, Q)$
- *if* $\omega_0 > \pi$ *then*

$$u(x,t) = c(x,t) r^{\pi/\omega_0} \sin \frac{\pi\omega}{\omega_0} + u_1(x,t),$$

where $c \in V^2_{\pi/\omega_0}(-\gamma, Q)$ *and* $u_1 \in H_0^2(-\gamma, Q)$.

Proposition 3.4. *Let* $f_{t^k} \in L_2(-\gamma, \mathbb{R})$ *for* $0 \le k \le 3$ *and assume that* u *be a generalized solution of the problem* (3.16)-(3.17) *for* $n = 3$. *Then we have*

$$|u(x,t)| \le C r^{Im\lambda_1}, \; C \text{ be a constant.}$$

Proposition 3.5. *Assume that* $f_{t^k} \in L_2(-\gamma, \mathbb{R})$ *for* $0 \le k \le 3$ *and* u *be a generalized solution of the problem* (3.16)-(3.17) *for* $n > 3$. *We have* $u \in H_0^2(-\gamma, Q)$.

CONCLUSION

RESULTS

This thesis has studied the boundary value problem without initial condition for Schrödinger systems in domain with conical points. The main results are:

- To construct a suitable definition of generalized solutions of the problem, then give the unique solvability of this problem in weighted Sobolev space $\overset{\circ}{H}{}^{m,0}(-\gamma, Q)$ and show a priori estimate of the solution.
- To investigate the smoothness of the generalized solutions of the problems. It is showed that the regularity with respect to time variable depends only on the right hand side while the regularity with respect to both time and space variables depend both the right hand side and the smoothness of the domain Ω.
- To establish several results on the asymptotic behavior of solutions near singular points. We weaken the assumptions on the eigenvalues of the associated spectrum problem (see condition (H)) but we still obtain asymptotic formulas of solutions.

The techniques and methods used in the thesis can be applied for other boundary value problems without initial conditions in domains with conical points.

RECOMMENDATION

Some open problems are:

- Study problems without initial conditions for others equations and systems.
- Study problems in domains with singular points.
- Study problems in weighted Sobolev spaces with L_p - norm for $1 \leq p \neq 2$.

THE WORKS RELATED TO THE THESIS

1) N. M. Hung, N. T. Lien (2013), "On the solvability of the boundary value problem without initial condition for Schrödinger systems in infinite cylinders", *Boundary Value Problems*, Vol. 2013, Article ID 2013:156, pp. 1-9.

2) N. M. Hung, N. T. Lien (2014), "On the regularity of solutions of the boundary value problem without initial condition for Schrö dinger systems in domain with conical points", *Boundary Value Problems*, Vol. 2014, Article ID 2014:181, pp. 1-12.

3) N. M. Hung, N. T. Lien (2016), "On the asymptotic formulas of solutions to the boundary value problem without initial condition for Schrödinger systems in domain with conical points", *Nonlinear Anal.*, Vol. 130, pp. 18-30.